

Life Span of Solutions for a Semilinear Heat Equation with Initial Data Non-Rarefied at ∞ *

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Abstract

We study the Cauchy problem for a semilinear heat equation with initial data non-rarefied at ∞ . Our interest lies in the discussion of the effect of the non-rarefied factors on the life span of solutions, and some sharp estimates on the life span is established.

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1 Introduction

Consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & (x, t) \in \mathbb{R}^n \times (0, +\infty), \\ u(x, 0) = \phi(x), & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where $p > 1$, and ϕ is a non-negative, bounded and continuous function in \mathbb{R}^n , which is not identically equal to zero.

It is well known that the solution $u(x, t)$ of (1) may blow up in finite time T^* , that is $\lim_{t \rightarrow T^*-} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} = \infty$, see [4, 7]. The finite time T^* is popularly called to be the *life span*, which depends heavily on the exponent p and the properties of the initial datum, and its study has arisen much attention during recent years, see for example [8, 6, 9, 10, 12] and references therein.

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It is worthy of mentioning that the properties of the initial data in some neighbourhood of ∞ are shown to be crucial factors affecting the life span of solutions. In this respect, for $\phi(x) \equiv \lambda\psi(x)$ with $\psi(x) \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\psi(x) \geq 0$, and $\psi(x)$ is positive in some neighbourhood of ∞ , the asymptotic behaviour of the life span T_λ seems to be interesting. Indeed, Lee and Ni [8] proved that if $\liminf_{|x| \rightarrow \infty} \psi(x) = k > 0$, then $C_1\lambda^{1-p} \leq T_\lambda \leq C_2\lambda^{1-p}$ for some positive constants C_1 and C_2 which are independent of λ ; while Gui and Wang [6] further showed that if $\lim_{|x| \rightarrow \infty} \psi(x) = k$, then $\lim_{\lambda \rightarrow 0^+} \lambda^{p-1}T_\lambda = \frac{1}{p-1}k^{1-p}$. More recently, Yamauchi [18] has found that as long as the initial datum $\phi(x)$ is positive in some **conic neighbourhood** of ∞ , the solution does blow up in a finite time T^* , and some elegant estimates on the life span are given.

The purpose of the present paper is to characterize the role of rarefaction properties of the initial data at ∞ , that is the effect on the life span of solutions of (1). We will show that as long as ∞ is not rarefied, although it is permitted for initial data to have zeros in any conic neighbourhood of ∞ , the solutions will blow up definitely. Of course, we are interested in estimating on the life span of solutions from the point of view of density analysis.

Before stating our main results we should recall or introduce some notations and conceptions. We begin with the definition of rarefaction point (see [15, p. 162]): A point x_0 is a *point of rarefaction of the set E* if

$$\lim_{r \rightarrow 0^+} \frac{\text{mes}(B(x_0, r) \cap E)}{\text{mes}(B(x_0, r))} = 0,$$

where $\text{mes}(F)$ is the Lebesgue measure of the set F , and $B(x, r)$ is the ball centered at x with radius r . It is reasonable to say that ∞ is a *point of rarefaction of the set E* if

$$\lim_{r \rightarrow +\infty} \frac{\text{mes}(B(0, r) \cap E)}{\text{mes}(B(0, r))} = 0.$$

Alternatively, ∞ is called to be a *rarefaction point of a non-negative function ϕ* if for any $\alpha > 0$ it is a point of rarefaction of the set $\{x \mid \phi(x) \geq \alpha\}$. We have the following theorem.

Theorem 1.1. *If ∞ is not a rarefaction point of the initial datum ϕ , then the solution of (1) blows up in finite time T^* with*

$$T^* \leq \frac{1}{p-1} \inf_{\alpha > 0} \left(\alpha D(\alpha) \right)^{1-p}, \quad (2)$$

where

$$D(\alpha) := \limsup_{r \rightarrow +\infty} \frac{\text{mes}(\{x \mid \phi(x) \geq \alpha\} \cap B(0, r))}{\text{mes}(B(0, r))}. \quad (3)$$

In what follows, we do not intend to give a proof for the above theorem, but prefer to present a stronger version of Theorem 1.1 in the following settings.

Let $\alpha, r > 0$, denote

$$D(\alpha; r) := \sup_{x \in \mathbb{R}^n} \frac{\text{mes}(B(x, r) \cap \{y \mid \phi(y) \geq \alpha\})}{\text{mes}(B(x, r))},$$

and define

$$\overline{D}(\alpha) := \limsup_{r \rightarrow +\infty} D(\alpha; r). \quad (4)$$

We are now in a position to present the main result of the following theorem.

Theorem 1.2. *Suppose that there exists $\alpha > 0$ such that $\overline{D}(\alpha) > 0$. Then the solution of (1) blows up in finite time T^* with*

$$T^* \leq \frac{1}{p-1} \left(\alpha \overline{D}(\alpha) \right)^{1-p}. \quad (5)$$

We shall give some comments on Theorem 1.2. First, by the definition of $\overline{D}(\alpha)$ and $D(\alpha)$, Theorem 1.2 implies Theorem 1.1 directly. Second, by a simple comparison argument we obtain the lower bound of the life span:

$$\frac{1}{p-1} \|\phi\|_{L^\infty(\mathbb{R}^n)}^{1-p} \leq T^*.$$

Thus Theorem 1.2 can show that the minimal time blow-up occurs¹ for some initial data (for the details see the example in Section 3). Finally, we point out that to prove Theorem 1.2 we take good advantage of basic properties of the heat kernel, so we believe that the argument is general and can be applied to similar problems posed on manifolds, e.g., the hyperbolic space, see Remark 2.2 and Remark 2.3.

This paper is organized as follows. In Section 2 we give the proof of Theorem 1.2. Subsequently, we show that Theorem 1.2 implies the main results of Yamauchi [18], in Section 3.

2 Proof of Theorem 1.2

Before proving Theorem 1.2 we first introduce the basic properties of the heat kernel in \mathbb{R}^n , and give a lemma on the life span of the solutions of (1).

The solution of (1) can be written as (see [3, p.51, (17)]):

$$u(x, t) = \int_{\mathbb{R}^n} g(x, y, t) \phi(y) dy + \int_0^t \left(\int_{\mathbb{R}^n} g(x, y, t-s) |u|^{p-1} u(y, s) dy \right) ds.$$

¹If the blow-up time of the solution of the problem (1) is the same as that of the ode: $u' = u^p$ with the initial value $u(0) = \|\phi\|_{L^\infty(\mathbb{R}^n)}$, we say that the minimal time blow-up occurs, for more details see Yamauchi [18] and references therein.

Here g is the heat kernel in \mathbb{R}^n , which is a function of the distance of $x, y \in \mathbb{R}^n$ and the time t , that is,

$$g(x, y, t) = k(d(x, y), t). \quad (6)$$

We summarize some elementary properties of the heat kernel in the following lemma.

Lemma 2.1 ([5, Section 1.3 and 2.7]). *Let $x, y, z \in \mathbb{R}^n$, and $s, t > 0$.*

- (i) $g(x, y, t) = g(y, x, t)$;
- (ii) $g(Tx, Ty, t) = g(x, y, t)$, where T is an isometry in \mathbb{R}^n ;
- (iii) *Semigroup property*

$$\int_{\mathbb{R}^n} g(x, y, t)g(y, z, s) dy = g(x, z, s + t); \quad (7)$$

- (iv) *Conservation of probability*

$$\int_{\mathbb{R}^n} g(x, y, t) dy = 1. \quad (8)$$

Remark 2.1. In \mathbb{R}^n it is well known that $g(x, y, t) = \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{|x-y|^2}{4t})$. In this paper we take T to be the translation $T(x) = x + x_0$ for fixed $x_0 \in \mathbb{R}^n$.

Remark 2.2. In Lemma 2.1 the properties (i) – (iii) are standard for the heat kernel and (iv) is satisfied for the manifold that is complete and with Ricci curvature bounded below (see [2, Theorem 5.2.6]). The property (6), used only to derive the uniform estimate (22), is satisfied for the homogeneous spaces. Thus the proofs of Theorem 2 depend only on the basic properties of the heat kernel, which *do not* depend on the explicit expression of the heat kernel. Therefore the proofs can be extended to the problems posed on other Riemannian manifolds, e.g., on the hyperbolic space².

Now we shall give an a priori estimate on the life span, which was essentially introduced by Weissler [17] for proving the blow-up of the non-trivial positive solutions of (1) in the critical case, see also [7, Chapter 5]. For the convenience of the reader, we prefer to represent it here.

Lemma 2.2. *Let $p > 1$, and $\phi \geq 0$ in $L^\infty(\mathbb{R}^n)$ is not identically zero. Suppose that u is a solution of the problem (1) with the initial value ϕ in $[0, T^*)$. Then*

$$\frac{1}{p-1} \left(\sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} g(z, y, t) \phi(y) dy \right)^{1-p} \geq t \quad \text{for } t \in (0, T^*). \quad (9)$$

In particular, we have the following upper bound estimate on the life span T^ :*

$$T^* \leq \sup \left\{ T > 0 \mid \frac{1}{p-1} \left(\sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} g(z, y, t) \phi(y) dy \right)^{1-p} \geq t \quad \text{for } t \in (0, T) \right\}.$$

²For blow-up problems of semilinear heat equations on the hyperbolic space, see the recent works of Bandle, Pozio and Tesei [1] and Wang and Yin [16].

Proof. Take $z \in \mathbb{R}^n$ and fix $0 < t < T^*$. Let $\bar{t} \in [0, t]$. Since $\phi(x) \geq 0$, by the comparison principle it follows that $u(x, t) \geq 0$. Thus we have

$$u(x, \bar{t}) = \int_{\mathbb{R}^n} g(x, y, \bar{t}) \phi(y) dy + \int_0^{\bar{t}} \int_{\mathbb{R}^n} g(x, y, \bar{t} - s) u^p(y, s) dy ds. \quad (10)$$

Multiplying (10) by $g(x, z, t - \bar{t})$ and integrating with respect to x over \mathbb{R}^n , we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} g(x, z, t - \bar{t}) u(x, \bar{t}) dx &= \int_{\mathbb{R}^n} g(x, z, t - \bar{t}) \int_{\mathbb{R}^n} g(x, y, \bar{t}) \phi(y) dy dx \\ &\quad + \int_{\mathbb{R}^n} g(x, z, t - \bar{t}) \int_0^{\bar{t}} \int_{\mathbb{R}^n} g(x, y, \bar{t} - s) u^p(y, s) dy ds dx. \end{aligned}$$

By (i) and (iii) in Lemma 2.1, and by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} g(z, x, t - \bar{t}) u(x, \bar{t}) dx &= \int_{\mathbb{R}^n} g(z, y, t) \phi(y) dy \\ &\quad + \int_0^{\bar{t}} \int_{\mathbb{R}^n} g(z, y, t - s) u^p(y, s) dy ds. \end{aligned}$$

Since $g(z, y, t - s) \geq 0$ and $\int_{\mathbb{R}^n} g(z, y, t - s) dy = 1$, by Jensen's inequality, the above equation implies

$$\begin{aligned} \int_{\mathbb{R}^n} g(z, x, t - \bar{t}) u(x, \bar{t}) dx &\geq \int_{\mathbb{R}^n} g(z, y, t) \phi(y) dy \\ &\quad + \int_0^{\bar{t}} \left(\int_{\mathbb{R}^n} g(z, y, t - s) u(y, s) dy \right)^p ds. \quad (11) \end{aligned}$$

Denote the right hand side of (11) by $G(\bar{t})$:

$$G(\bar{t}) := \int_{\mathbb{R}^n} g(z, y, t) \phi(y) dy + \int_0^{\bar{t}} \left(\int_{\mathbb{R}^n} g(z, y, t - s) u(y, s) dy \right)^p ds.$$

We have $G(t) > 0$ for $t \in [0, T^*)$ and

$$G(0) = \int_{\mathbb{R}^n} g(z, y, t) \phi(y) dy. \quad (12)$$

Differentiating $G(\bar{t})$ with respect to \bar{t} , by (11) we have the inequality

$$G'(\bar{t}) = \left(\int_{\mathbb{R}^n} g(z, y, t - \bar{t}) u(y, \bar{t}) dy \right)^p \geq G^p(\bar{t});$$

that is

$$G^{-p}(\bar{t}) G'(\bar{t}) \geq 1. \quad (13)$$

Integrating (13) with respect to \bar{t} over $[0, t]$, we obtain

$$\frac{1}{1-p} \left(G^{1-p}(t) - G^{1-p}(0) \right) \geq t.$$

Thus

$$\frac{1}{p-1} G^{1-p}(0) \geq t, \quad (14)$$

since $p > 1$. Then the lemma follows from (12) and (14). \square

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. By the assumption of the theorem, for any $\varepsilon > 0$ there exist $z_k \in \mathbb{R}^n$ and $r_k > 0$ for $k = 1, 2, \dots$, such that

$$\lim_{k \rightarrow \infty} r_k = +\infty, \quad (15)$$

and

$$\frac{\text{mes}(B(z_k, r_k) \cap \{y \mid \phi(y) \geq \alpha\})}{\omega_n r_k^n} \geq \overline{D}(\alpha) - \varepsilon, \quad (16)$$

where ω_n is the volume of n -dimensional unit ball.

Take $\bar{r}_k = \sqrt{r_k}$. We claim the following lemma, which will be proved in the end of this section.

Lemma 2.3. *For any $\delta \in (0, 1)$, there exists $K \in \mathbb{N}$ such that for $k > K$*

$$\begin{aligned} & \sup_{x \in B(z_k, r_k - \bar{r}_k)} \int_{B(z_k, r_k)} \mathbb{1}_{B(x, \bar{r}_k)}(y) g(x, y, T^*) \phi(y) \, dy \\ & \geq \frac{1-\delta}{(r_k - \bar{r}_k)^n} \left[\alpha (\overline{D}(\alpha) - \varepsilon) r_k^n - \|\phi\|_{L^\infty(\mathbb{R}^n)} (r_k^n - (r_k - 2\bar{r}_k)^n) \right]. \end{aligned} \quad (17)$$

Here $\mathbb{1}_E$ is the characteristic function of the set E .

Lemma 2.3 implies that for $k > K$

$$\begin{aligned} & \sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} g(z, y, T^*) \phi(y) \, dy \geq \sup_{x \in B(z_k, r_k - \bar{r}_k)} \int_{B(z_k, r_k)} \mathbb{1}_{B(x, \bar{r}_k)}(y) g(x, y, T^*) \phi(y) \, dy \\ & \geq \frac{1-\delta}{(r_k - \bar{r}_k)^n} \left[\alpha (\overline{D}(\alpha) - \varepsilon) r_k^n - \|\phi\|_{L^\infty(\mathbb{R}^n)} (r_k^n - (r_k - 2\bar{r}_k)^n) \right]. \end{aligned} \quad (18)$$

Let $k \rightarrow \infty$. (15) and (18) show

$$\sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} g(z, y, T^*) \phi(y) \, dy \geq (1-\delta) \alpha (\overline{D}(\alpha) - \varepsilon);$$

and since δ and ε were arbitrary, we obtain

$$\sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} g(z, y, T^*) \phi(y) \, dy \geq \alpha \overline{D}(\alpha). \quad (19)$$

Now the theorem follows from (19) and Lemma 2.2 immediately. \square

Finally, we should give the proof of Lemma 2.3.

Proof of Lemma 2.3. It is sufficient to show that there exists K satisfies the following property: for $k > K$ there exists $x_k \in B(z_k, r_k - \bar{r}_k)$ such that

$$\int_{B(z_k, r_k)} \mathbb{1}_{B(x_k, \bar{r}_k)}(y) g(x_k, y, T^*) \phi(y) \, dy \geq \frac{1 - \delta}{(r_k - \bar{r}_k)^n} \left[\alpha(\overline{D}(\alpha) - \varepsilon) r_k^n - \|\phi\|_{L^\infty(\mathbb{R}^n)} (r_k^n - (r_k - 2\bar{r}_k)^n) \right]. \quad (20)$$

For any $k = 1, 2, \dots$, we define a sequence of functions $F_k : B(z_k, r_k) \rightarrow \overline{\mathbb{R}^+}$ as

$$F_k(x) := \int_{B(z_k, r_k)} \mathbb{1}_{B(x, \bar{r}_k)}(y) g(x, y, T^*) \phi(y) \, dy.$$

Then F_k is a non-negative, continuous and bounded function. Consider the integral of $F_k(x)$ over $B(z_k, r_k - \bar{r}_k)$. By Fubini's theorem, we have

$$\begin{aligned} \int_{B(z_k, r_k - \bar{r}_k)} F_k(x) \, dx &= \int_{B(z_k, r_k)} \mathbb{1}_{B(z_k, r_k - \bar{r}_k)}(x) F_k(x) \, dx \\ &= \int_{B(z_k, r_k)} \mathbb{1}_{B(z_k, r_k - \bar{r}_k)}(x) \left(\int_{B(z_k, r_k)} \mathbb{1}_{B(x, \bar{r}_k)}(y) g(x, y, T^*) \phi(y) \, dy \right) \, dx \\ &= \int_{B(z_k, r_k)} \left(\int_{B(z_k, r_k - \bar{r}_k)} \mathbb{1}_{B(z_k, r_k - \bar{r}_k)}(x) \mathbb{1}_{B(x, \bar{r}_k)}(y) g(x, y, T^*) \, dx \right) \phi(y) \, dy \\ &= \int_{B(z_k, r_k)} I_k(y) \phi(y) \, dy, \end{aligned} \quad (21)$$

where

$$I_k(y) := \int_{B(z_k, r_k)} \mathbb{1}_{B(z_k, r_k - \bar{r}_k)}(x) \mathbb{1}_{B(x, \bar{r}_k)}(y) g(x, y, T^*) \, dx.$$

Since $g(x, y, t)$ is a function of the distance between x and y for fixed t ; and since

$$\int_{\mathbb{R}^n} g(x, y, T^*) \, dx = 1 \quad \text{for } y \in \mathbb{R}^n,$$

it follows that for fixed $\delta > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{R}^n} \mathbb{1}_{B(y, \bar{r}_k)}(x) g(x, y, T^*) \, dx = \int_{B(y, \bar{r}_k)} g(x, y, T^*) \, dx \geq 1 - \delta \quad (22)$$

for any $y \in \mathbb{R}^n$ and any $\bar{r}_k > R$. Since $\bar{r}_k = \sqrt{r_k}$ and $r_k \rightarrow \infty$ as $k \rightarrow \infty$, there exists K such that

$$\bar{r}_k > R \quad \text{for } k > K.$$

Thus by (22) we obtain

$$\int_{\mathbb{R}^n} \mathbb{1}_{B(y, \bar{r}_k)}(x) g(x, y, T^*) \, dx \geq 1 - \delta \quad \text{for } k > K. \quad (23)$$

In the following proof we shall always take $k > K$. Moreover, for $y \in B(z_k, r_k - 2\bar{r}_k)$, it is easily seen that

$$\mathbb{1}_{B(y, \bar{r}_k)}(x) = \mathbb{1}_{B(x, \bar{r}_k)}(y) \quad \text{and} \quad B(y, \bar{r}_k) \subset B(z_k, r_k - \bar{r}_k). \quad (24)$$

Then by (24) and (22) we obtain, for $y \in B(z_k, r_k - 2\bar{r}_k)$,

$$\begin{aligned} I_k(y) &= \int_{B(z_k, r_k - \bar{r}_k)} \mathbb{1}_{B(x, \bar{r}_k)}(y) g(x, y, T^*) \, dx \\ &= \int_{B(z_k, r_k - \bar{r}_k)} \mathbb{1}_{B(y, \bar{r}_k)}(x) g(x, y, T^*) \, dx \\ &= \int_{\mathbb{R}^n} \mathbb{1}_{B(y, \bar{r}_k)}(x) g(x, y, T^*) \, dx \geq 1 - \delta. \end{aligned}$$

Thus by the above inequality and the definition of I_k we have

$$I_k(y) \geq \begin{cases} 1 - \delta, & y \in B(z_k, r_k - 2\bar{r}_k), \\ 0, & y \in B(z_k, r_k) \setminus B(z_k, r_k - 2\bar{r}_k). \end{cases} \quad (25)$$

By (21) and (25) we obtain

$$\begin{aligned} \int_{B(z_k, r_k - \bar{r}_k)} F_k(x) \, dx &= \int_{B(z_k, r_k)} I_k(y) \phi(y) \, dy \\ &= \int_{B(z_k, r_k - 2\bar{r}_k)} I_k(y) \phi(y) \, dy + \int_{B(z_k, r_k) \setminus B(z_k, r_k - 2\bar{r}_k)} I_k(y) \phi(y) \, dy \\ &\geq \int_{B(z_k, r_k - 2\bar{r}_k)} I_k(y) \phi(y) \, dy \geq (1 - \delta) \int_{B(z_k, r_k - 2\bar{r}_k)} \phi(y) \, dy, \end{aligned}$$

which, together with the continuity of $F_k(x)$, implies that there exists $x_k \in B(z_k, r_k - \bar{r}_k)$ such that

$$F_k(x_k) \geq \frac{(1 - \delta) \int_{B(z_k, r_k - 2\bar{r}_k)} \phi(y) \, dy}{\omega_n (r_k - \bar{r}_k)^n}. \quad (26)$$

Since (16) implies that

$$\begin{aligned} \int_{B(z_k, r_k)} \phi(y) \, dy &\geq \alpha \int_{B(z_k, r_k) \cap \{y \mid \phi(y) \geq \alpha\}} dy \\ &\geq \alpha \, \text{mes}(B(z_k, r_k) \cap \{x \mid \phi(x) \geq \alpha\}) \geq \alpha (\bar{D}(\alpha) - \varepsilon) \omega_n r_k^n, \end{aligned}$$

we obtain

$$\begin{aligned} \int_{B(z_k, r_k - 2\bar{r}_k)} \phi(y) \, dy &= \int_{B(z_k, r_k)} \phi(y) \, dy - \int_{B(z_k, r_k) \setminus B(z_k, r_k - 2\bar{r}_k)} \phi(y) \, dy \\ &\geq \alpha (\bar{D}(\alpha) - \varepsilon) \omega_n r_k^n - \|\phi\|_{L^\infty(\mathbb{R}^n)} \omega_n (r_k^n - (r_k - 2\bar{r}_k)^n). \end{aligned} \quad (27)$$

Then by (26) and (27)

$$F_k(x_k) \geq \frac{1-\delta}{(r_k - \bar{r}_k)^n} \left[\alpha(\bar{D}(\alpha) - \varepsilon) r_k^n - \|\phi\|_{L^\infty(\mathbb{R}^n)} (r_k^n - (r_k - 2\bar{r}_k)^n) \right],$$

which is exactly (20) by the definition of F_k . This completes the proof. \square

Remark 2.3. Since in the hyperbolic space \mathbb{H}^n the volume of the ball of radius r is $\sigma^{n-1}(\mathbb{S}^{n-1}) \int_0^r \sinh^{n-1} \eta d\eta$, where $\sigma^{n-1}(\mathbb{S}^{n-1})$ is the surface area of the $n-1$ Euclidean sphere of radius 1, see [13, p. 79]; and since the heat kernel in the hyperbolic space satisfies all the corresponding properties in Lemma 2.1 and (6), we can modify the above proofs to adapt the similar problem posted in the hyperbolic space, say, replacing the Δ in (1) with the Laplace-Beltrami operator in \mathbb{H}^n .

3 Proving Theorem 3.1 by Theorem 1.2

In this section we shall show that Theorem 1.2 implies the previous results of Yamauchi in [18]. To state Yamauchi's results precisely we recall some notations in [18]. For $\xi' \in \mathbb{S}^{n-1}$ and $\delta \in (0, \sqrt{2})$, we set conic neighbourhood $\Gamma_{\xi'}(\delta)$:

$$\Gamma_{\xi'}(\delta) = \{\eta \in \mathbb{R}^n \setminus \{0\} \mid |\xi' - \frac{\eta}{|\eta|}| < \delta\},$$

and set

$$S_{\xi'}(\delta) = \Gamma_{\xi'}(\delta) \cap \mathbb{S}^{n-1}. \quad (28)$$

Define

$$\phi_\infty(x') = \liminf_{r \rightarrow +\infty} \phi(rx') \quad \text{for } x' \in \mathbb{S}^{n-1}.$$

Yamauchi proved the following theorem.

Theorem 3.1 ([18, Theorem 1 and Theorem 2]). *Let $n \geq 2$. Assume that there exist $\xi' \in \mathbb{S}^{n-1}$ and $\delta > 0$ such that $\text{ess inf}_{x' \in S_{\xi'}(\delta)} \phi_\infty(x') > 0$. Then the classical solution for (1) blows up in finite time, and the blow-up time is estimated as*

$$T^* \leq \frac{1}{p-1} \left(\text{ess inf}_{x' \in S_{\xi'}(\delta)} \phi_\infty(x') \right)^{1-p}. \quad (29)$$

Let $n = 1$. Assume that $\max\{\liminf_{x \rightarrow +\infty} \phi(x), \liminf_{x \rightarrow -\infty} \phi(x)\} > 0$. Then the classical solution for (1) blows up in finite time, and the blow-up time is estimated as

$$T^* \leq \frac{1}{p-1} \left(\max\{\liminf_{x \rightarrow +\infty} \phi(x), \liminf_{x \rightarrow -\infty} \phi(x)\} \right)^{1-p}. \quad (30)$$

It is easily seen that Theorem 3.1 implies the upper bounded estimate of Gui and Wang in [6] immediately; that is, if $\lim_{|x| \rightarrow \infty} \psi(x) = k$, then $\lambda^{p-1} T_\lambda \leq$

$\frac{1}{p-1}k^{1-p}$. However, Theorem 3.1 can not be applied to some simple cases, which can be illustrated by the following example.

Take a sequence $\{a_k\}_{k=1}^\infty$ by $a_k = k!$. We see that $\lim_{k \rightarrow \infty} \frac{a_k}{a_{k+1}} = 0$. Using the sequence $\{a_k\}_{k=1}^\infty$ we can construct a function $\Phi \in C(\mathbb{R}^n)$, which satisfies $0 \leq \Phi(x) \leq 1$ and

$$\Phi(x) = \begin{cases} 0, & |x| \in [a_{2k-1} + \frac{1}{4}, a_{2k} - \frac{1}{4}], \\ 1, & |x| \in [a_{2k}, a_{2k+1}]. \end{cases}$$

By the definition of $D(\alpha)$ in (3) it follows that

$$D(1) \geq \limsup_{k \rightarrow +\infty} \frac{\text{mes}(\{x \mid \phi(x) \geq 1\} \cap B(0, a_{2k+1}))}{\text{mes}(B(0, a_{2k+1}))} \geq \limsup_{k \rightarrow +\infty} \frac{a_{2k+1}^n - a_{2k}^n}{a_{2k+1}^n} = 1.$$

Thus for the initial datum $u_0(x) = \Phi(x)$, by Theorem 1.1 the life span T^* of the solution u can be estimated by

$$T^* \leq \frac{1}{p-1} (1D(1))^{1-p} = \frac{1}{p-1}.$$

Since $v(x, t) = (1 - (p-1)t)^{\frac{1}{p-1}}$, which blows up at $T = \frac{1}{p-1}$, is an upper solution of (1) with $v(x, 0) \geq u(x, 0)$, by the comparison principle it follows that $T^* \geq T = \frac{1}{p-1}$. Thus $T^* = \frac{1}{p-1}$, which shows that the minimal time blow-up occurs for the initial datum $\Phi(x)$.

Now we use Theorem 1.2 to prove Theorem 3.1.

Proof of Theorem 3.1. We shall prove the case $n = 1$ and $n \geq 2$, respectively.

(i) $n = 1$. Let us assume that

$$A = \liminf_{x \rightarrow +\infty} \phi(x) \geq \liminf_{x \rightarrow -\infty} \phi(x).$$

Then for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\phi(x) \geq A - \varepsilon \quad \text{for } x > R.$$

Hence $\overline{D}(A - \varepsilon) = 1$. By Theorem 1.2 we obtain

$$T^* \leq \frac{1}{p-1} (A - \varepsilon)^{1-p}.$$

Since $\varepsilon > 0$ was arbitrary, it follows

$$T^* \leq \frac{1}{p-1} A^{1-p}.$$

The proof for $n = 1$ is finished.

(ii) $n \geq 2$. Let $A = \text{ess inf}_{x' \in S_{\xi'}(\delta)} \phi_\infty(x')$. It is sufficient to show that, for any $\bar{R} > 0$, $0 < \tau < 1$ and $0 < \varepsilon < A$, there exists a ball $B(x, r)$ with $r > \bar{R}$ such that

$$\frac{\text{mes}\left(\{y \mid \phi(y) \geq A - \varepsilon\} \cap B(x, r)\right)}{\omega_n r^n} > 1 - \tau. \quad (31)$$

We shall show (31) in three steps.

First, we show that there exists $R_1 > 0$ such that

$$\frac{\sigma^{n-1}\left(\{x' \in \mathbb{S}^{n-1} \mid \phi(rx') \geq A - \varepsilon \text{ for } r > R_1\} \cap S_{\xi'}(\delta)\right)}{\sigma^{n-1}(S_{\xi'}(\delta))} > 1 - \tau, \quad (32)$$

where $\sigma^{n-1}(M)$ is the spherical measure for the measurable set $M \subset \mathbb{S}^{n-1}$ and $S_{\xi'}(\delta)$ is defined by (28).

We denote

$$S(R) := \{x' \in \mathbb{S}^{n-1} \mid \phi(rx') \geq A - \varepsilon \text{ for } r > R\},$$

and claim that the set

$$S(R) \cap S_{\xi'}(\delta)$$

is measurable on \mathbb{S}^{n-1} . Indeed, since the function $\phi(x)$ is continuous, we may view ϕ as a continuous function on $\mathbb{S}^{n-1} \times \overline{R^+}$, which implies that, for fixed $r > 0$, $\phi(rx')$ is continuous on \mathbb{S}^{n-1} . Then, for any $r > 0$, the set

$$\{x' \in \mathbb{S}^{n-1} \mid \phi(rx') \geq A - \varepsilon\}$$

is closed on \mathbb{S}^{n-1} . Since

$$\begin{aligned} S(R) &= \{x' \in \mathbb{S}^{n-1} \mid \phi(rx') \geq A - \varepsilon \text{ for } r > R\} \\ &= \bigcap_{r > R} \{x' \in \mathbb{S}^{n-1} \mid \phi(rx') \geq A - \varepsilon\}, \end{aligned}$$

it follows that the set $S(R)$ is closed. Hence $S(R) \cap S_{\xi'}(\delta)$ is measurable on \mathbb{S}^{n-1} , and the claim is proved.

By the assumption of Theorem 3.1, we see that

$$S_{\xi'}(\delta) = \bigcup_{k=1,2,\dots} (S(k) \cap S_{\xi'}(\delta)).$$

Noticing that

$$S(k_1) \cap S_{\xi'}(\delta) \subset S(k_2) \cap S_{\xi'}(\delta) \text{ for } k_1 < k_2,$$

by the standard convergence theorem for a sequence of measurable sets (see, e.g., [14, Theorem 11.3]), we obtain

$$\sigma^{n-1}(S_{\xi'}(\delta)) = \lim_{k \rightarrow \infty} \sigma^{n-1}(S(k) \cap S_{\xi'}(\delta)).$$

Thus, for any $0 < \tau < 1$, there exists $K \in \mathbb{N}$ such that

$$\frac{\sigma^{n-1}(S(k) \cap S_{\xi'}(\delta))}{\sigma^{n-1}(S_{\xi'}(\delta))} > 1 - \tau \quad \text{for } k \geq K.$$

We can take $R_1 = K$ to ensure (32).

Next, we show that there exists a ball $B(x_0, r_0)$ such that

$$B(x_0, r_0) \subset \{x \in \mathbb{R}^n \mid R_1 < |x| < R_1 + 1 \text{ and } \frac{x}{|x|} \in S_{\xi'}(\delta)\}, \quad (33)$$

and

$$\frac{\text{mes}(\{x \in B(x_0, r_0) \mid \frac{x}{|x|} \in S(R_1)\})}{\text{mes}(B(x_0, r_0))} \geq 1 - \tau. \quad (34)$$

For this we need the following covering lemma, where $\overline{B(x, r)}$ is the closure of the ball $B(x, r)$.

Lemma 3.1 ([11, Theorem 2.7] Besicovitch Covering Theorem). *Suppose μ is a Borel measure on \mathbb{R}^n , $E \subset \mathbb{R}^n$, $\mu(E) < \infty$, \mathcal{F} is a collection of non-trivial closed balls, and $\inf\{r \mid \overline{B(x, r)} \in \mathcal{F}\} = 0$ for all $x \in E$. Then there is a countable disjoint sub-collection of \mathcal{F} that covers μ almost all of E .*

We choose μ to be the Lebesgue measure on \mathbb{R}^n ,

$$E = \{x \in \Omega \mid \frac{x}{|x|} \in S(R_1)\},$$

and take

$$\mathcal{F} = \{\overline{B(x, r)} \subset \Omega \mid x \in E, r > 0\},$$

where $\Omega = \{x \in \mathbb{R}^n \mid R_1 < |x| < R_1 + 1, \frac{x}{|x|} \in S_{\xi'}(\delta)\}$ is an open set of \mathbb{R}^n by the definition of $S_{\xi'}(\delta)$. Noticing that the map $\mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{S}^{n-1}$, defined by $x \rightarrow \frac{x}{|x|}$, is continuous and $S(R_1)$ is a closed set of \mathbb{S}^{n-1} , we see that E is a measurable set of \mathbb{R}^n . Then by Lemma 3.1 we can take a sequence of balls $\{\overline{B(x_i, r_i)}\}_{i=1}^\infty \subset \mathcal{F}$ such that

$$\overline{B(x_i, r_i)} \cap \overline{B(x_j, r_j)} = \emptyset \quad \text{for } i \neq j, \quad (35)$$

and

$$\text{mes}(E \setminus \bigcup_i \overline{B(x_i, r_i)}) = 0. \quad (36)$$

Thus by (32) and the definition of E we obtain

$$\begin{aligned} \text{mes}(E) &= \int_{R_1}^{R_1+1} r^{n-1} \int_{S_{\xi'}(\delta)} \mathbb{1}_{S(R_1) \cap S_{\xi'}(\delta)}(\xi) d\xi dr \\ &\geq (1 - \tau) \int_{R_1}^{R_1+1} r^{n-1} \int_{S_{\xi'}(\delta)} d\xi dr \\ &= (1 - \tau) \text{mes}(\Omega). \end{aligned} \quad (37)$$

By (35), (36) and the definition of \mathcal{F} , we have

$$\text{mes}(E) = \text{mes}\left(\bigcup_i \left(\overline{B(x_i, r_i)} \cap E\right)\right) = \sum_i \text{mes}\left(\overline{B(x_i, r_i)} \cap E\right) \quad (38)$$

and

$$\text{mes}(\Omega) \geq \text{mes}\left(\bigcup_i \overline{B(x_i, r_i)}\right) = \sum_i \text{mes}\left(\overline{B(x_i, r_i)}\right). \quad (39)$$

Inserting (38) and (39) into (37), we obtain

$$\sum_i \text{mes}\left(\overline{B(x_i, r_i)} \cap E\right) \geq (1 - \tau) \sum_i \text{mes}\left(\overline{B(x_i, r_i)}\right).$$

Then we can get a ball $B(x_0, r_0)$ in the sequence $\{B(x_i, r_i)\}_{i=1}^\infty$ satisfying (33) and (34).

Finally, we complete the proof by scaling the ball $B(x_0, r_0)$. We claim that for $x \in \mathbb{R}^n$, $r > 0$ and $\lambda > 1$, it follows that

$$\frac{\text{mes}\left(\{y \in \mathbb{R}^n \mid \phi(y) \geq A - \varepsilon\} \cap B(\lambda x, \lambda r)\right)}{\text{mes}\left(B(\lambda x, \lambda r)\right)} \geq \frac{\text{mes}\left(B(x, r) \cap E\right)}{\text{mes}\left(B(x, r)\right)}. \quad (40)$$

The claim can be seen by the following argument. By the definitions of E and $S(R_1)$ it follows that if $z \in B(x, r) \cap E$ then $\phi(\lambda z) \geq A - \varepsilon$ and $\lambda z \in B(\lambda x, \lambda r)$ for $\lambda > 1$. Thus

$$\{y \in \mathbb{R}^n \mid \phi(y) \geq A - \varepsilon\} \cap B(\lambda x, \lambda r) \supset \{\lambda z \mid z \in B(x, r) \cap E\}.$$

Hence

$$\begin{aligned} & \text{mes}\left(\{y \in \mathbb{R}^n \mid \phi(y) \geq A - \varepsilon\} \cap B(\lambda x, \lambda r)\right) \\ & \geq \text{mes}\left(\{\lambda z \mid z \in B(x, r) \cap E\}\right) = \lambda^n \text{mes}\left(B(x, r) \cap E\right), \end{aligned}$$

and the claim follows. By (33) we see that $r_0 < 1$, thus $\frac{\bar{R}+1}{r_0} > 1$. Scaling the ball $B(x_0, r_0)$ with $\lambda = \frac{\bar{R}+1}{r_0}$, by (40) and (34) we obtain

$$\frac{\text{mes}\left(\{y \in \mathbb{R}^n \mid \phi(y) \geq A - \varepsilon\} \cap B\left(\frac{\bar{R}+1}{r_0}x_0, \bar{R}+1\right)\right)}{\text{mes}\left(B\left(\frac{\bar{R}+1}{r_0}x_0, \bar{R}+1\right)\right)} \geq \frac{\text{mes}\left(B(x_0, r_0) \cap E\right)}{\text{mes}\left(B(x_0, r_0)\right)} \geq 1 - \tau.$$

Thus the ball $B\left(\frac{\bar{R}+1}{r_0}x_0, \bar{R}+1\right)$ satisfies (31), and the proof is complete. \square

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